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J. Differential Equations 201 (2004) 324–350
Journal of Differential Equations
http://www.elsevier.com/locate/jde

An infinite-dimensional linking theorem and applications

Martin Schechter[a,1] and Wenming Zou[a,b,*,2]

[a] *Department of Mathematics, University of California, Irvine, CA 92697-3875, USA*
[b] *Department of Mathematical Sciences, Tsinghua University, Beijing 10084, China*

Received March 17, 2003; revised August 28, 2003

Abstract

In this paper, we establish a new infinite-dimensional linking theorem without (PS)-type assumptions. The new theorem needs a weaker linking geometry and produces bounded (PS) sequences. The abstract result will be applied to the study of the existence of solutions of the strongly indefinite partial differential systems. For the first application, we consider the system

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \Delta v = v & \text{in } \Omega, \end{cases} \begin{cases} \partial u/\partial\eta = H_v(x,u,v) & \text{on } \partial\Omega, \\ \partial v/\partial\eta = H_u(x,u,v) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary, $\frac{\partial}{\partial\eta}$ is the outer normal derivative, $H: \partial\Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a positive $\mathbf{C}^1$-function. One nontrivial solution is obtained. The second application, we will solve the eigenvalue problem of the system

$$\begin{cases} \mathscr{A}v = -f(x,v,w), \\ \mathscr{B}w = \beta g(x,v,w), \quad \beta > 0, \end{cases}$$

where $\mathscr{A}$, $\mathscr{B}$ are self-adjoint operators on $L^2(\Omega)$, $\Omega \subset \mathbf{R}^N$ *is not necessarily bounded*; f, g are Carathéodory functions on $\Omega \times \mathbf{R}^2$. We get infinitely many solutions. We deal with asymptotically linear cases for both systems.

MSC: 35J65; 35J20; 35J55

*Corresponding author. Fax: +1-949-824-7993.
E-mail addresses: mschecht@math.uci.edu (M. Schechter), wzou@math.uci.edu (W. Zou).
[1] Supported in part by a NSF Grant.
[2] Supported by NSFC Grant (GP10001019). Current address: Department of Mathematics, University of California, Irvine, CA 92697.

0022-0396/$ - see front matter
doi:10.1016/j.jde.2004.03.006

1. Introduction

Let E be a real Hilbert space with inner product $(\cdot,\cdot)$, norm $||\cdot||$ and decomposition $E := E^- \oplus E^+$, with both E^+ and E^- infinite-dimensional. Let $G_\lambda : E \to \mathbf{R}$ be a $\mathbf{C}^1$-functional having the structure

$$G_\lambda(u) = (L_\lambda u, u) + H(u), \quad \lambda \in [1,2],$$

where $L_\lambda : E \to E$ is a linear, bounded, self-adjoint operator; H' is compact. Let A, B be two subsets of E such that A links B in the sense of Definition 2.1 below. If G_λ satisfies a weaker linking geometry (i.e., $\sup_A G_\lambda \leqslant \inf_B G_\lambda$) for each $\lambda \in [1,2]$, we shall be concerned with how to obtain a critical point without the usual Palais–Smale compactness condition. Loosely speaking, we shall show, for almost all $\lambda \in [1,2]$, that G_λ has a critical point. To this, we will adopt the monotonicity method developed in [J1,J2] (see also [GJ,JT], an earlier idea can be found in [St1,St2]) and the ideas of linking due to [S1,S2]. The abstract result will be used to study the existence of nontrivial solutions to the elliptic system

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \Delta v = v & \text{in } \Omega \end{cases}$$

with nonlinear boundary conditions

$$\begin{cases} \frac{\partial u}{\partial \eta} = H_v(x,u,v) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \eta} = H_u(x,u,v) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary, $\frac{\partial}{\partial \eta}$ is the outer normal derivative. As a second application we study the system with one parameter

$$\begin{cases} \mathscr{A} v = -f(x,v,w), \\ \mathscr{B} w = \beta g(x,v,w), \quad \beta > 0, \end{cases}$$

where $\mathscr{A} \geqslant \beta_0 > 0$, $\mathscr{B} \geqslant \mu_0 > 0$ are self-adjoint operators on $L^2(\Omega)$, $\Omega \subset \mathbf{R}^N$ *is not necessarily bounded*, $f(x,s,t)$ and $g(x,s,t)$ are Carathéodory functions on $\Omega \times \mathbf{R}^2$. We prove the existence of infinitely many solutions. Both systems are strongly indefinite. We consider asymptotically linear cases.

2. An infinite-dimensional linking theorem

Let E be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $||\cdot||$. A subset $A \subset E$ links a subset $B \subset E$ if $A \cap B = \emptyset$, and for every $\Gamma \in \Phi$ there is a $t \in (0,1]$ such that $\Gamma(t)A \cap B \neq \emptyset$, where $\Phi = \Phi(E)$ is the set of mappings $\Gamma(t) \in \mathbf{C}([0,1] \times E, E)$ such that

(1) $\Gamma(0) = I$;

(2) there is a $u_0 \in E$ depending on Γ such that $\Gamma(t)u \to u_0$ as $t \to 1$ uniformly on bounded subsets of E;
(3) for each $t \in [0,1)$, $\Gamma(t)$ is a homeomorphism of E onto itself and $\Gamma^{-1}(t) \in \mathbf{C}([0,1) \times E, E)$.

We assume that E has an orthogonal decomposition $E = E^+ \oplus E^-$. Let Λ^* be the set of those continuous mappings

$$\mu(t,u) = \gamma(t,u)u + c(t,u), \quad u \in E, \ 0 \leqslant t < 1$$

such that

(a) $\gamma(\cdot,\cdot) \in \mathbf{C}([0,1) \times E, B(E))$, where $B(E)$ denotes the set of the bounded linear operators from E to E;
(b) for each $t \in [0,1), v \in E, \gamma(t,v)$ is a linear homeomorphism of E onto E and E^- onto E^-;
(c) for each $t \in [0,1), \gamma(t,\cdot) \in K(E, B(E))$;
(d) $c(t,u) \in \mathbf{C}([0,1) \times E, E)$;
(e) for each $t \in [0,1), c(t,\cdot)$ is compact on E.

We let Φ^* denote the set of those $\Gamma \in \Phi$ such that Γ and Γ^{-1} are both in Λ^*. Then Φ^* is not empty since it contains the map $\Gamma(t)u = (1-t)u + tu_0$.

Definition 2.1. A set $A \subset E$ links a set $B \subset E$ with respect to Φ^* if $A \cap B = \emptyset$ and for each $\Gamma \in \Phi^*$, there is a $t \in [0,1]$ such that $\Gamma(t)A \cap B \neq \emptyset$.

Evidently, if A links B with respect to Φ, it links B with respect to Φ^*. However, there are sets that link with respect to Φ^* but do not link with respect to Φ (cf. [S2]).

The concept of linking plays a fundamental role in critical point theory. Now let's say a few words about the usual linking (cf., e.g., [BBF,BeR,R]): Let B be a closed subset of a Banach space E and let Q be a submanifold of E with relative boundary ∂Q. For $A \subset E$, let C_A denotes the set of all $\phi \in \mathbf{C}(E,E)$ that leave points of A fixed. Then $A = \partial Q$ links B if $A \cap B = \emptyset$ and $\phi(\bar{Q}) \cap B \neq \emptyset, \phi \in C_A$. More examples can be found in [R]. In [ST], the authors introduced a concept of linking: A set $A \subset E$ links a set $B \subset E$ with respect to Φ if $A \cap B = \emptyset$ and for each $\Gamma \in \Phi$, there is a $t \in [0,1]$ such that $\Gamma(t)A \cap B \neq \emptyset$. This linking avoided some drawbacks. It is possible for $A = \partial(E^- \cap \bar{B}_R)$ to link another set if either $\dim E^- < \infty$ or $\dim E^+ < \infty$. But $A = \partial(E^- \cap \bar{B}_R)$ does not link E^+ if both $\dim E^- = \infty$ and $\dim E^+ = \infty$. Definition 2.1 is a generalization of the above linking. It fits perfectly to some stronger indefinite functionals. Moreover, under this definition, we permit a weaker *linking geometry* for the functional to possess and we get more information on the location of critical point. Here there are two examples.

Example 2.1. $E^- \cap \partial B_R$ links E^+ with respect to Φ^* for every $R > 0$ in the sense of Definition 2.1.

Example 2.2. Let v_0 be a vector in $E^+\backslash\{0\}$. Let $0<\rho<R$,

$$Q \coloneqq \{u = v + sv_0 : v\in E^-, s\geqslant 0, ||u||\leqslant R\},$$

$$A \coloneqq \partial Q, \quad B \coloneqq \{u\in E^+ : ||u|| = \rho\}.$$

Then A links B with respect to Φ^* in the sense of Definition 2.1.

On the applications of the linking theorems in [BBF,BeR,R,S2,St2], the Palais–Smale condition and its variants (cf. e.g. [BBF]) play an important role. A very recent paper due to [J1] proved that the mountain pass theorem provides a bounded Palias–Smale sequence by monotonicity methods. A generalization was given in [SZ]. Some ideas of the present paper comes from [J1].

Let $m>1$ and $\{G_\lambda\}$ be a family of $\mathbf{C}^1(E,\mathbf{R})$ functional having the following form:

$$G_\lambda(u) \coloneqq \frac{1}{2}(L_\lambda u, u) + H(u), \quad \lambda\in[1,m],$$

and satisfying

(C_1) There are two linear, bounded, self-adjoint operators $L^{(1)}, L^{(2)} : E\to E$ such that

$$L_\lambda = \lambda L^{(1)} - L^{(2)}, \quad \lambda\in[1,m],$$

where $(L^{(1)}u,u)\geqslant 0$, for all $u\in E$ and either $(L^{(1)}u,u)\to\infty$ or $|(L^{(2)}u,u) - H(u)|\to\infty$ as $||u||\to\infty$.

(C_2) H' is compact.

(C_3) L_λ is invertible for all $\lambda\in[1,m]$.

The main result is the following theorem.

Theorem 2.1. *Let E^- be an invariant subspace of L_λ and assume that $A,B\subset E$, A links B with respect to Φ^*, A is bounded. Assume also that*

$$a_0(\lambda) \coloneqq \sup_A G_\lambda \leqslant b_0(\lambda) \coloneqq \inf_B G_\lambda, \quad \forall\lambda\in[1,m],$$

and

$$a_\lambda \coloneqq \inf_{\Gamma\in\Phi^*} \sup_{u\in A, s\in[0,1]} G_\lambda(\Gamma(s)u) < \infty.$$

Then for almost all $\lambda\in[1,m]$, there exists a bounded *sequence $\{z_n\}$ such that*

$$G_\lambda{}'(z_n)\to 0; \quad G_\lambda(z_n)\to a_\lambda \quad \text{as } n\to\infty.$$

Furthermore, if $a_\lambda = b_0(\lambda)$, we have that $\operatorname{dist}(z_n,B)\to 0$ *as $n\to\infty$. Finally, G_λ has a critical value for almost all $\lambda\in[1,m]$ and the critical point lies in B if $a_\lambda = b_0(\lambda)$.*

Proof. Since the map $\lambda \mapsto a_\lambda$ is nondecreasing, the derivative $a_\lambda' := \frac{\partial a_\lambda}{\partial \lambda}$ exists for almost every $\lambda \in [1, m]$. Here, we use a technique due to [J1]. From now on, we consider those λ where a_λ' exists. For fixed $\lambda \in [1, m)$, let $\lambda_n \in (\lambda, m)$ be a sequence such that $\lambda_n \to \lambda$ as $n \to \infty$. Then there exists an $\bar{n}(\lambda)$ such that

$$a_\lambda' - 1 \leqslant \frac{a_{\lambda_n} - a_\lambda}{\lambda_n - \lambda} \leqslant a_\lambda' + 1, \quad \text{for } n \geqslant \bar{n}(\lambda). \tag{2.1}$$

We show, for almost all $\lambda \in [1, m]$, that there exist $\Gamma_n \in \Phi^*, k_0 := k_0(\lambda) > 0$ such that

$$||\Gamma_n(s)u|| \leqslant k_0 \quad \text{whenever} \quad G_\lambda(\Gamma_n(s)u) \geqslant a_\lambda - (\lambda_n - \lambda). \tag{2.2}$$

In fact, by the definition of a_{λ_n}, there exists $\Gamma_n \in \Phi^*$ such that

$$\begin{aligned} \sup_{u \in A, s \in [0,1]} G_\lambda(\Gamma_n(s)u) \leqslant & \sup_{u \in A, s \in [0,1]} G_{\lambda_n}(\Gamma_n(s)u) \\ & \leqslant a_{\lambda_n} + (\lambda_n - \lambda) \\ & \leqslant (a_\lambda' + 1)(\lambda_n - \lambda) + a_\lambda + (\lambda_n - \lambda) \\ & \leqslant a_\lambda + (a_\lambda' + 2)(\lambda_n - \lambda). \end{aligned} \tag{2.3}$$

If $G_\lambda(\Gamma_n(s)u) \geqslant a_\lambda - (\lambda_n - \lambda)$ for some $u \in A, s \in [0, 1]$, then by assumptions (C_1), (2.1) and (2.3), we have that

$$\begin{aligned} (L^{(1)}\Gamma_n(s)u, \Gamma_n(s)u) &= \frac{G_{\lambda_n}(\Gamma_n(s)u) - G_\lambda(\Gamma_n(s)u)}{\lambda_n - \lambda} \\ &\leqslant \frac{a_{\lambda_n} + (\lambda_n - \lambda) - a_\lambda + (\lambda_n - \lambda)}{\lambda_n - \lambda} \\ &\leqslant a_\lambda' + 3. \end{aligned} \tag{2.4}$$

Therefore

$$\begin{aligned} & (L^{(2)}\Gamma_n(s)u, \Gamma_n(s)u) - H(u) \\ & \quad = \lambda_n(L^{(1)}\Gamma_n(s)u, \Gamma_n(s)u) - G_{\lambda_n}(\Gamma_n(s)u) \\ & \quad \leqslant \lambda_n(a_\lambda' + 3) - G_\lambda(\Gamma_n(s)u) \\ & \quad \leqslant \lambda_n(a_\lambda' + 3) - a_\lambda + (\lambda_n - \lambda) \\ & \quad \leqslant m(a_\lambda' + 3) - a_\lambda + m. \end{aligned} \tag{2.5}$$

On the other hand, by assumption (C_1), (2.1) and (2.3),

$$\begin{aligned}
&(L^{(2)}\Gamma_n(s)u, \Gamma_n(s)u) - H(u) \\
&\quad = \lambda_n(L^{(1)}\Gamma_n(s)u, \Gamma_n(s)u) - G_{\lambda_n}(\Gamma_n(s)u) \\
&\quad \geqslant -G_{\lambda_n}(\Gamma_n(s)u) \\
&\quad \geqslant -(a_{\lambda_n} + (\lambda_n - \lambda)) \\
&\quad \geqslant -(a_\lambda + (\lambda_n - \lambda)(a_\lambda{}' + 2)) \\
&\quad \geqslant -a_\lambda - m|a_\lambda{}' + 2|.
\end{aligned} \tag{2.6}$$

Combining (2.4)–(2.6) and (C_1), we observe that there exists a $k_0(\lambda) := k_0$ depending only on λ, such that $||\Gamma_n(s)u|| \leqslant k_0$.

Case 1: We first deal with the case of $b_0(\lambda) > a_0(\lambda)$. Inspired by [J1] (and [GJ,J2,JT]]), for any $\varepsilon > 0$, let

$$M_\varepsilon(\lambda) := \{u \in E : ||u|| \leqslant k_0 + 3, |G_\lambda(u) - a_\lambda| \leqslant \varepsilon\}. \tag{2.7}$$

Then by the definition of a_λ, (2.2) and (2.3), we observe that $M_\varepsilon(\lambda) \neq \emptyset$. Now we claim, for ε small enough, that $\inf\{||G_\lambda{}'(u)|| : u \in M_\varepsilon(\lambda)\} = 0$.

By negation, we assume that there is a $\varepsilon_0 > 0$ such that

$$||G_\lambda{}'(u)|| \geqslant \varepsilon_0, \quad \forall u \in M_{\varepsilon_0}(\lambda). \tag{2.8}$$

Without loss of generality, we assume $\varepsilon_0 < \min\{2/9, (b_0(\lambda) - a_0(\lambda))/2\}$. Choose n large enough, so that

$$(|a_\lambda{}'| + 3)(\lambda_n - \lambda) < 3\varepsilon_0^2/4. \tag{2.9}$$

Since H' is compact, by Proposition A.23 of [R, p. 87] (see also [BE]), there exists an operator $W : E \to E$ which is compact, locally Lipschitz continuous such that

$$||H'(u) - W(u)|| \leqslant \varepsilon_0/4, \quad \forall u \in E. \tag{2.10}$$

Set

$$V_\lambda(u) := L_\lambda(u) + W(u), \quad \forall u \in E. \tag{2.11}$$

Then V_λ is locally Lipschitz continuous and, by (2.8), (2.10) and (2.11), it has the following properties:

$$||V_\lambda(u)|| \geqslant ||G_\lambda{}'(u)|| - \varepsilon_0/4 \geqslant 3\varepsilon_0/4, \quad \forall u \in M_{\varepsilon_0}(\lambda); \tag{2.12}$$

$$
\begin{aligned}
(G_\lambda{}'(u), V_\lambda(u)) &\geqslant ||G_\lambda{}'(u)||^2 - ||G_\lambda(u)|| ||H'(u) - W(u)|| \\
&\geqslant ||G_\lambda{}'(u)||^2 - \varepsilon_0 ||G_\lambda(u)||/4 \geqslant 3||G_\lambda{}'(u)||^2/4, \quad \forall u \in M_{\varepsilon_0}(\lambda); \qquad (2.13)
\end{aligned}
$$

$$||V_\lambda(u)|| \leqslant ||G_\lambda{}'(u)|| + \varepsilon_0/4 \leqslant 5||G_\lambda{}'(u)||/4, \quad \forall u \in M_{\varepsilon_0}(\lambda). \qquad (2.14)$$

Let

$$
\begin{aligned}
A_1 &:= \{u \in E : ||u|| \geqslant k_0 + 2\} \cup \{u \in E : G_\lambda(u) \leqslant a_\lambda - \varepsilon_0/3\} \\
&\quad \cup \{u \in E : G_\lambda(u) \geqslant a_\lambda + \varepsilon_0/3\}; \qquad (2.15)
\end{aligned}
$$

$$B_1 := \{u \in E : ||u|| \leqslant k_0 + 1, |G_\lambda(u) - a_\lambda| \leqslant \varepsilon_0/4\}. \qquad (2.16)$$

Then $B_1 \subset M_{\varepsilon_0}(\lambda), A_1 \cap B_1 = \emptyset$. Let $\psi(s) := 1$ for $s \in [0, 1]$ and $\psi(s) := 1/s$ for $s \geqslant 1$. Set

$$\xi(u) := \frac{\operatorname{dist}(u, A_1)}{\operatorname{dist}(u, B_1) + \operatorname{dist}(u, A_1)}$$

and consider the vector field

$$V_\lambda^*(u) := \xi(u)\psi(||V_\lambda(u)||) V_\lambda(u). \qquad (2.17)$$

Since $u \notin A_1$ implies that $u \in M_{\varepsilon_0}(\lambda)$, if we combine (2.13), (2.14) and (2.17), we can easily check that

$$(G_\lambda{}'(u), V_\lambda^*(u)) \geqslant 0 \quad \text{and} \quad ||V_\lambda^*(u)|| \leqslant 1 \quad \text{for all } u \in E. \qquad (2.18)$$

Furthermore, for any $u \in B_1 \subset M_{\varepsilon_0}(\lambda)$, by (2.8), (2.13) and (2.14), we see that

$$(G_\lambda{}'(u), V_\lambda^*(u)) \geqslant 3\varepsilon_0^2/4. \qquad (2.19)$$

Consider the following initial value problem:

$$\frac{d\eta_\lambda(t, u)}{dt} = -V_\lambda^*(\eta_\lambda), \quad \eta_\lambda(0, u) = u.$$

It is well known that there is a unique solution $\eta_\lambda \in \mathbf{C}([0, 1] \times E, E)$. By (2.11), (2.17) and (2.18), we see that (cf. [R, Proposition A. 18])

$$\frac{\partial G_\lambda(\eta_\lambda(t, u))}{\partial t} \leqslant 0, \quad \forall u \in E, \ \ t \in [0, 1], \qquad (2.20)$$

$$\eta_\lambda(t, u) = \exp\left(\left(\int_0^t -\xi(\eta_\lambda(s, u))\psi(||V_\lambda(\eta_\lambda(s, u))||)\, ds\right) L_\lambda\right) u + K_\lambda(t, u), \qquad (2.21)$$

where K_λ is a compact map. Define

$$\eta_\lambda^*(t)u := \eta_\lambda(t, \Gamma_n(t)u), \quad \forall t \in [0, 1], \ \ \forall u \in E. \qquad (2.22)$$

Evidently, $\eta_\lambda^* \in \Phi^*$.

For any $u \in A$, we consider two cases:

- If $G_\lambda(\Gamma_n(s)u) \leqslant a_\lambda - (\lambda_n - \lambda)$, then by (2.20) and (2.22)

$$G_\lambda(\eta_\lambda^*(s)u) \leqslant G_\lambda(\Gamma_n(s)u) \leqslant a_\lambda - (\lambda_n - \lambda). \tag{2.23}$$

- If $G_\lambda(\Gamma_n(s)u) > a_\lambda - (\lambda_n - \lambda)$, then $||\Gamma_n(s)u|| \leqslant k_0$. On the other hand, by (2.3) and (2.9),

$$\sup_{u \in A, s \in [0,1]} G_\lambda(\Gamma_n(s)u) \leqslant a_\lambda + (2 + a_\lambda{}')(\lambda_n - \lambda) \leqslant a_\lambda + \varepsilon_0/6. \tag{2.24}$$

It follows that $\Gamma_n(s)u \in M_{\varepsilon_0}(\lambda)$. Furthermore, we assume that $G_\lambda(\eta_\lambda^*(s)u) > a_\lambda - (\lambda_n - \lambda)$ (Otherwise, we get the desired contradiction by combining this with (2.23)). For $t \in [0,1]$, by (2.20) and (2.24),

$$\begin{aligned} a_\lambda - (\lambda_n - \lambda) < {} & G_\lambda(\eta_\lambda^*(s)u) \\ & \leqslant G_\lambda(\eta_\lambda(s, \Gamma_n(s)u)) \\ & \leqslant G_\lambda(\Gamma_n(s)u) \\ & \leqslant a_\lambda + \varepsilon_0/6. \end{aligned} \tag{2.25}$$

By (2.18),

$$\begin{aligned} & ||\eta_\lambda(t, \Gamma_n(s)u) - \Gamma_n(s)u|| \\ & \quad = \left|\left| \int_0^t \frac{d\eta_\lambda(t, \Gamma_n(s)u)}{dt}\, dt \right|\right| \\ & \quad \leqslant \int_0^t ||V_\lambda^*(\eta_\lambda(t, \Gamma_n(s)u))||\, dt \\ & \quad \leqslant t, \end{aligned}$$

hence,

$$||\eta_\lambda(t, \Gamma_n(s)u)|| \leqslant t + ||\Gamma_n(s)u|| \leqslant 1 + k_0 \quad \text{for } t, s \in [0,1]. \tag{2.26}$$

Combining (2.25) and (2.26), we see that $\eta_\lambda(t, \Gamma_n(s)u) \in B$ (cf.(2.16)) for $t, s \in [0,1]$. Recalling (2.19), we have that

$$\begin{aligned} & G_\lambda(\eta_\lambda(t, \Gamma_n(s)u)) - G_\lambda(\Gamma_n(s)u) \\ & \quad = \int_0^t \frac{dG_\lambda(\eta_\lambda(t, \Gamma_n(s)u))}{dt}\, dt \\ & \quad = -\int_0^t (G_\lambda{}'(\eta_\lambda(t, \Gamma_n(s)u)), V_\lambda^*(\eta_\lambda(t, \Gamma_n(s)u)))\, dt \\ & \quad \leqslant -\frac{3\varepsilon_0^2}{4}. \end{aligned} \tag{2.27}$$

By (2.24)–(2.27),

$$\begin{aligned}
&G_\lambda(\eta_\lambda^*(s)u)\\
&\quad\leqslant G_\lambda(\Gamma_n(s)u) - \frac{3\varepsilon_0^2}{4}\\
&\quad\leqslant a_\lambda + (2 + a_\lambda{}')(\lambda_n - \lambda) - \frac{3\varepsilon_0^2}{4}\\
&\quad\leqslant a_\lambda - (\lambda_n - \lambda).
\end{aligned} \tag{2.28}$$

However, (2.23) and (2.28) contradict the definition of a_λ. This completes the proof for this case.

Case 2: We consider the case $b_0(\lambda) = a_0(\lambda)$.

Since A is bounded, $d_A := \max\{||u|| : u\in A\} < \infty$. For $\varepsilon > 0, T > 0$, we define

$$\bar{Q}(\varepsilon, T, \lambda) := \{u\in E : ||u|| \leqslant k_0(\lambda) + 4 + d_A, |G_\lambda(u) - a(\lambda)| \leqslant 3\varepsilon, d(u, B) \leqslant 4T\}. \tag{2.29}$$

We claim that $\bar{Q}(\varepsilon, T, \lambda) \neq \emptyset$. By (2.3), we choose n large enough such that

$$\sup_{s\in[0,1], u\in A} G_\lambda(\Gamma_n(s)u) \leqslant \sup_{s\in[0,1], u\in A} G_{\lambda_n}(\Gamma_n(s)u) \leqslant a_\lambda + 3\varepsilon. \tag{2.30}$$

Since A links B, there exists $(s_0, u_0)\in[0,1]\times A$ such that $\Gamma_n(s_0)u_0\in B$. Hence $\operatorname{dist}(\Gamma_n(s_0)u_0, B) = 0$ and

$$G_\lambda(\Gamma_n(s_0)u_0) \geqslant b_0(\lambda) = \inf_B G_\lambda = a_\lambda > a_\lambda - (\lambda_n - \lambda) \geqslant a_\lambda - 3\varepsilon. \tag{2.31}$$

By the arguments given at the beginning of the proof, $||\Gamma_n(s_0)u_0|| \leqslant k_0$. Hence, $\Gamma_n(s_0)u_0\in\bar{Q}(\varepsilon, T, \lambda)$.

In the spirit of [J1], it suffices to show that

$$\inf\{||G_\lambda{}'(u)|| : u\in\bar{Q}(\varepsilon, T, \lambda)\} = 0 \quad \text{for } \varepsilon, T \text{ small enough.} \tag{2.32}$$

If not, there exists $\delta > 0, \varepsilon_1 > 0, T_1\in(0,1)$ such that

$$||G_\lambda{}'(u)|| \geqslant 3\delta \quad \text{for } u\in\bar{Q}(\varepsilon_1, T_1, \lambda). \tag{2.33}$$

Without loss of generality, we assume that $\delta \leqslant \frac{1}{4}$. Let n be so large that $(|a_\lambda{}'| + 2)(\lambda_n - \lambda) < \varepsilon_1$ and $(\lambda_n - \lambda) < \delta T_1$. Similar to the first case, there exists an operator $W : E \to E$ which is compact, locally Lipschitz continuous such that

$$||H'(u) - W(u)|| \leqslant \delta, \quad \forall u\in E. \tag{2.34}$$

Set

$$\bar{V}_\lambda(u) := L_\lambda(u) + W(u), \quad \forall u\in E, \tag{2.35}$$

then $\bar{V}_\lambda$ is locally Lipschitz continuous and, by (2.34), (2.35) and (2.33), it has the following properties:

$$||\bar{V}_\lambda(u)|| \geqslant ||G_\lambda{}'(u)|| - \delta \geqslant 2\delta, \quad \forall u \in \bar{Q}(\varepsilon_1, T_1, \lambda), \tag{2.36}$$

$$\begin{aligned}(G_\lambda{}'(u), \bar{V}_\lambda(u)) &\geqslant ||G_\lambda{}'(u)||^2 - ||G_\lambda(u)|| ||H'(u) - W(u)|| \\ &\geqslant ||G_\lambda{}'(u)||^2 - \delta ||G_\lambda(u)|| \geqslant 2||G_\lambda{}'(u)||^2/3; \quad \forall u \in \bar{Q}(\varepsilon_1, T_1, \lambda),\end{aligned} \tag{2.37}$$

$$||V_\lambda(u)|| \leqslant ||G_\lambda{}'(u)|| + \delta \leqslant 4||G_\lambda{}'(u)||/3, \quad \forall u \in \bar{Q}(\varepsilon_1, T_1, \lambda). \tag{2.38}$$

Let $\psi(s) := 1$ for $s \in [0, 1]$, $\psi(s) := 1/s$ for $s \geqslant 1$ and

$$Y_\lambda(u) := \psi(||\bar{V}_\lambda(u)||)\bar{V}_\lambda(u).$$

Then it is easy to check that

$$(G_\lambda{}'(u), Y_\lambda(u)) \geqslant 6\delta^2, \quad \forall u \in \bar{Q}(\varepsilon_1, T_1, \lambda). \tag{2.39}$$

Define

$$Q_1 := \{u \in E : ||u|| \leqslant k_0 + 2 + d_A, |G_\lambda(u) - a_\lambda| \leqslant 2\varepsilon_1, \mathrm{dist}(u, B) \leqslant 3T_1\}. \tag{2.40}$$

Then $Q_1 \neq \emptyset$ and $Q_1 \subset \bar{Q}(\varepsilon_1, T_1, \lambda)$. Choose a Lipschitz continuous map γ from E into $[0, 1]$ which equals 1 on Q_1 and vanishes outside $\bar{Q}(\varepsilon_1, T_1, \lambda)$. Consider the following initial boundary value problem

$$\frac{d(\bar{\eta}_\lambda(t, u))}{dt} = -\gamma(\bar{\eta}_\lambda) Y_\lambda(\bar{\eta}_\lambda), \quad \bar{\eta}_\lambda(0, u) = u.$$

Let $\bar{\eta}_\lambda(t, u)$ be the unique continuous solution. Then we have that

$$\frac{dG_\lambda(\bar{\eta}_\lambda(t, u))}{dt} \leqslant -6\delta^2 \gamma(\bar{\eta}_\lambda(t, u)) \leqslant 0, \tag{2.41}$$

$$\bar{\eta}_\lambda(t, u) = \exp\left(\left(-\int_0^t \gamma(\bar{\eta}_\lambda(s, u))\, ds\right) L_\lambda\right) u + \tilde{K}_\lambda(t, u),$$

where $\tilde{K}_\lambda$ is a compact map.

Claim 1: $\bar{\eta}_\lambda(t, u) \notin B$ for all $t \in [0, T_1]$ and $u \in A$.

For $u \in A$, by (2.41), we have that

$$G_\lambda(\bar{\eta}_\lambda(t, u)) \leqslant G_\lambda(u) \leqslant a_0(\lambda) \leqslant b_0(\lambda) = a_\lambda, \quad \forall t \in [0, T_1] \tag{2.42}$$

and

$$G_\lambda(\bar{\eta}_\lambda(t,u)) = G_\lambda(u) + \int_0^t \frac{dG_\lambda(\bar{\eta}_\lambda(\sigma,u))}{d\sigma}\, d\sigma$$
$$\leqslant G_\lambda(u) - \int_0^t 6\delta^2 \gamma(\bar{\eta}_\lambda(\sigma,u))\, d\sigma \tag{2.43}$$

for all $t \in [0, T_1]$.

If Claim 1 is not true, then there is a $t_0 \in [0, T_1]$ such that $\bar{\eta}_\lambda(t_0, u) \in B$. Then $G_\lambda(\bar{\eta}_\lambda(t_0,u)) \geqslant a_\lambda = b_0(\lambda) = \inf_B G_\lambda$. By (2.41)–(2.42), we see that $\int_0^{t_0} 6\delta^2\gamma(\bar{\eta}_\lambda(\sigma,u))\, d\sigma = 0$. Hence, $\gamma(\bar{\eta}_\lambda(\sigma,u)) = 0$ for $\sigma \in [0,t_0]$, i.e., $\bar{\eta}_\lambda(\sigma,u) \notin Q_1 \forall \sigma \in [0,t_0]$. Therefore, one of the following three cases occurs:

$$||\bar{\eta}_\lambda(\sigma,u)|| > k_0 + 2 + d_A, \tag{2.44}$$

$$|G_\lambda(\bar{\eta}_\lambda(\sigma,u)) - a(\lambda)| > 2\varepsilon_1, \tag{2.45}$$

$$\mathrm{dist}(\bar{\eta}_\lambda(\sigma,u), B) > 3T_1. \tag{2.46}$$

Since

$$||\bar{\eta}_\lambda(\sigma,u) - \bar{\eta}_\lambda(\sigma',u)|| \leqslant |\sigma - \sigma'|, \tag{2.47}$$

we see that

$$||\bar{\eta}_\lambda(\sigma,u)|| \leqslant ||\bar{\eta}_\lambda(0,u)|| + T_1 \leqslant d_A + 1, \quad \forall \sigma \in [0,t_0].$$

Then (2.47) would not happen.

If (2.45) holds, then $G_\lambda(\bar{\eta}_\lambda(\sigma,u)) < a_\lambda - 2\varepsilon_1$. Hence, $\bar{\eta}_\lambda(\sigma,u) \notin B$. Evidently, (2.46) implies that $\bar{\eta}_\lambda(\sigma,u) \notin B$. Therefore, Claim 1 is true.

Claim 2: $\bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u) \notin B, \forall u \in A, s \in [1/2, 1]$.

For any fixed $u \in A$ and $s \in [1/2, 1]$, we divide the proof into two cases.

Case (i): If $\bar{\eta}_\lambda(\sigma, \Gamma_n(2s-1)u) \in Q_1$ for all $\sigma \in [0, T_1]$, by (2.39) and (2.41), we have that

$$\begin{aligned}
&G_\lambda(\bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u)) \\
&\quad = G_\lambda(\Gamma_n(2s-1)u) + \int_0^{T_1} \frac{dG_\lambda(\bar{\eta}_\lambda(\sigma,\Gamma_n(2s-1)u))}{d\sigma}\, d\sigma \\
&\quad \leqslant G_\lambda(\Gamma_n(2s-1)u) - \int_0^{T_1} 6\delta^2\gamma(\bar{\eta}_\lambda(\sigma,\Gamma_n(2s-1)u))\, d\sigma \\
&\quad = G_\lambda(\Gamma_n(2s-1)u) - 6\delta^2 T_1 \\
&\quad \leqslant a_\lambda - 6\delta^2 T_1 + (a_\lambda{}' + 2)(\lambda_n - \lambda)
\end{aligned}$$

which implies that $\bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u) \notin B$ since $a_\lambda = b_0(\lambda)$.

Case (ii): If there exists $t_0 \in [0, T_1]$ such that $\bar{\eta}_\lambda(t_0, \Gamma_n(2s-1)u) \notin Q_1$, then one of the following alternatives holds:

$$||\bar{\eta}_\lambda(t_0, \Gamma_n(2s-1)u)|| > k_0 + 2 + d_A, \tag{2.48}$$

$$|G_\lambda(\bar{\eta}_\lambda(t_0, \Gamma_n(2s-1)u)) - a_\lambda| > 2\varepsilon_1, \tag{2.49}$$

$$\mathrm{dist}(\bar{\eta}_\lambda(t_0, \Gamma_n(2s-1)u), B) > 3T_1. \tag{2.50}$$

Assume that (2.31) holds. If $\bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u) \in B$, then

$$b_0(\lambda) = a_\lambda \leqslant G_\lambda(\bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u) \leqslant G_\lambda(\Gamma_n(2s-1)u).$$

Then $||\Gamma_n(2s-1)u|| \leqslant k_0$. Furthermore, since

$$||\bar{\eta}_\lambda(t_0, \Gamma_n(2s-1)u) - \bar{\eta}_\lambda(0, \Gamma_n(2s-1)u)|| \leqslant t_0,$$

it follows that

$$||\bar{\eta}_\lambda(t_0, \Gamma_n(2s-1)u)|| \leqslant k_0 + t_0 \leqslant k_0 + 1,$$

which contradicts (2.48). Hence, $\bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u) \notin B$.

Assume that (2.48) holds. Note, by (2.6), that

$$\begin{aligned} G_\lambda(\bar{\eta}_\lambda(t_0, \Gamma_n(2s-1)u)) &\leqslant G_\lambda(\bar{\eta}_\lambda(0, \Gamma_n(2s-1)u)) \\ &= G_\lambda(\Gamma_n(2s-1)u) \\ &\leqslant a_\lambda + \varepsilon_1. \end{aligned}$$

Therefore, (2.49) implies that

$$G_\lambda(\bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u)) \leqslant G_\lambda(\bar{\eta}_\lambda(t_0, \Gamma_n(2s-1)u)) \leqslant a(\lambda) - 2\varepsilon_1.$$

It follows that $\bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u) \notin B$ since $a(\lambda) = b_0(\lambda)$.

Assume (2.50) holds. Note that $||\bar{\eta}_\lambda(t, u) - \bar{\eta}_\lambda(t', u)|| \leqslant |t - t'|$. It follows that

$$||\bar{\eta}_\lambda(t, \Gamma_n(2s-1)u) - w|| \geqslant ||\bar{\eta}_\lambda(t_0, \Gamma_n(2s-1)u) - w|| - |t - t_0|$$

for all $w \in B, t \in [0, T_1]$. Hence, $\mathrm{dist}(\bar{\eta}_\lambda(t, \Gamma_n(2s-1)u), B) \geqslant 2T_1$ for all $t \in [0, T_1]$. In particular,

$$\bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u) \notin B.$$

This completes the proof of Claim 2.

In order to get the final contradiction, we define

$$\Gamma_1^*(s,u) := \begin{cases} \bar{\eta}_\lambda(2sT_1, u), & 0 \leqslant s \leqslant 1/2, \\ \bar{\eta}_\lambda(T_1, \Gamma_n(2s-1)u), & 1/2 \leqslant s \leqslant 1. \end{cases}$$

Then $\Gamma_1^* \in \Phi^*$. However, by Claims 1 and 2, $\Gamma_1^*(s, A) \cap B = \emptyset$ for all $s \in [0, 1]$. We get the final contradiction. $\square$

3. Application (I)

In this section, we consider the following elliptic system:

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \Delta v = v & \text{in } \Omega \end{cases} \tag{3.1}$$

with nonlinear boundary condition

$$\begin{cases} \frac{\partial u}{\partial \eta} = H_v(x,u,v) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \eta} = H_u(x,u,v) & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where Ω is a bounded domain of $\mathbf{R}^N$ with smooth boundary, $\frac{\partial}{\partial \eta}$ is the outer normal derivative.

The existence of solutions for nonlinear elliptic systems has created a great deal of interest in recent years, in particular when the nonlinearities appear as a source in the systems with Dirichlet boundary conditions (see e.g. [BAF,BOF,CFM,F2,FI1,-FI2,FF,FM,FMI,FMT,HV,SI1,SI2], [Z] and the references cited therein). In [BPR1], the authors study the variational structure of (3.1)–(3.2) and get a nontrivial solution for superlinear case by using a minimax theorem due to [F1]. We also note, in [BR] by using fixed-point theory, that the authors consider (3.1)–(3.2) without variational assumptions.

In this section, we consider the asymptotically linear case. To the best of our knowledge, for this case, no result have appeared before. Since we may get bounded (PS) sequence, the assumptions on H are rather weaker than those used in other asymptotically linear problem (cf. e.g. [S1,SZ]). This is easy for superlinear case with the aid of Ambrosetti–Rabinowitz type condition. However, for superlinear case without Ambrosetti–Rabinowitz type assumption and for asymptotically linear case, the problems become more complicated. In this paper, by virtue of the new linking theorem built in Section 2, we shall solve these problems easily. We like to mention two related papers [J1,SZ]. In [J1], the author's parameter depending Mountain Pass theorem is a finite-dimensional linking and cannot be applied to (3.1)–(3.2). In [SZ], the authors need a stronger linking geometry, that is, the "sup" and "inf" must be separated by a positive constant. This is much more demanding on the hypotheses of H around zero. Other classical theories (for instance, [BeR]) need a Palais–Smale compactness condition or its variant, Cérami's compactness condition. Theorem 2.1 of this paper can avoid these drawbacks.

We first recall some basic facts. Let $A : D(A) \subset L^2(\Omega) \times L^2(\partial\Omega) \to L^2(\Omega) \times L^2(\partial\Omega)$ be the operator defined by

$$A(z, z|_{\partial\Omega}) \coloneqq \left(-\Delta z + z, \frac{\partial z}{\partial \eta}\right),$$

where $D(A) \coloneqq \{(z, z|_{\partial\Omega}) : z \in H^2(\Omega)\}$. Then $D(A)$ is dense in $L^2(\Omega) \times L^2(\partial\Omega)$ and A is positive and symmetric, A^{-1} is bounded and compact. Therefore, there exists a sequence of eigenvalues $\{\bar{\lambda}_n\} \subset \mathbf{R}^+$ of A with eigenfunctions $(\phi_n, \psi_n) \in L^2(\Omega) \times L^2(\partial\Omega)$ (cf. [BPR1, p. 4, 14]) such that

$$\begin{cases} 0 < \bar{\lambda}_1 \leqslant \bar{\lambda}_2 \leqslant \cdots \leqslant \bar{\lambda}_n \cdots \nearrow \infty, \\ \phi_n \in H^2(\Omega), \phi_n|_{\partial\Omega} = \psi_n; \quad \phi_1 > 0 \text{ on } \bar{\Omega}. \end{cases}$$

We know that $\{\lambda_n\} \subset \mathbf{R}^+$ and $(\phi_n, \psi_n \coloneqq \phi_n|_{\partial\Omega}) \in L^2(\Omega) \times L^2(\partial\Omega)$ are the solutions of the eigenvalue problem

$$\begin{cases} -\Delta\phi_n + \phi_n = \bar{\lambda}_n \phi_n & \text{in } \Omega, \\ \frac{\partial \phi_n}{\partial \eta} = \bar{\lambda}_n \phi_n & \text{on } \partial\Omega. \end{cases}$$

For $u \coloneqq \sum_{n=1}^{\infty} a_n(\phi_n, \psi_n) \in L^2(\Omega) \times L^2(\partial\Omega)$ and $s \in (0,1)$, define the operator $A^s : D(A^s) \to L^2(\Omega) \times L^2(\partial\Omega)$ with $A^s u \coloneqq \sum_{n=1}^{\infty} \bar{\lambda}_n^s a_n(\phi_n, \psi_n)$. Let $E^s \coloneqq D(A^s)$, which is a Hilbert space with the inner product and norm

$$(z, w)_{E^s} = \langle A^s z, A^s w \rangle, \quad ||z||_{E^s} = (z, z)_{E^s}^{1/2},$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(\Omega) \times L^2(\partial\Omega)$ given by

$$\langle (u, v), (\phi, \psi) \rangle = \int_\Omega u\phi + \int_{\partial\Omega} v\psi.$$

Since we are only concerned with the asymptotically linear case, we may choose $s = \frac{1}{2}$ next.

Now, we make the following assumptions:

(T_1) $0 \leqslant H(x, u, v) \leqslant c(1 + |u|^\alpha + |v|^\beta)$, $\forall (x, u, v) \in \partial\Omega \times \mathbf{R} \times \mathbf{R}$, where

$$\alpha, \beta \in \left(1, \frac{2(N-1)}{N-2}\right] \text{ if } N > 2, \quad \alpha, \beta \in (1, +\infty) \quad \text{if } N = 1, 2.$$

Particularly,

$$H(x, u, u) \leqslant \bar{\lambda}_1 (1 + |u|^2), \quad \forall (x, u) \in \partial\Omega \times \mathbf{R}.$$

(T_2) Either $H_u(x,0,0)\neq 0$ or $H_v(x,0,0)\neq 0$ for $x\in\partial\Omega$. Moreover,

$$\frac{H_u(x,u,v)}{\sqrt{|u|^2+|v|^2}}\to 0, \quad \frac{H_v(x,u,v)}{\sqrt{|u|^2+|v|^2}}\to 0,$$

as $|u|^2+|v|^2\to\infty$ uniformly for $x\in\partial\Omega$.

The main result is

Theorem 3.1. *Assume that* $H:\partial\Omega\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ *satisfies* (T_1)–(T_2). *Then elliptic system* (3.1)–(3.2) *has a nontrivial weak solution* $(u,v)\in W^{2,2}(\Omega)\times W^{2,2}(\Omega)$ *satisfying*

$$\int_\Omega(\nabla u\nabla\phi+u\phi)-\int_{\partial\Omega}H_v(x,u,v)\phi=0,$$

$$\int_\Omega(\nabla v\nabla\phi+v\phi)-\int_{\partial\Omega}H_u(x,u,v)\phi=0$$

for all smooth ϕ.

Let $E:=E^{1/2}\times E^{1/2}$. Then E is a Hilbert space with norm $||\cdot||_E$ induced by the inner product

$$((u,v),(\phi,\psi))_E=\langle A^{1/2}u,A^{1/2}\phi\rangle+\langle A^{1/2}v,A^{1/2}\psi\rangle. \tag{3.3}$$

Moreover, E has a natural orthogonal decomposition $E:=E^+\oplus E^-$, where

$$E^+:=\{(u,u):u\in E^{1/2}\},\quad E^-:=\{(u,-u):u\in E^{1/2}\}.$$

By this splitting, the projections $P^\pm:E\to E^\pm$ are given by (cf. [BPR1, Lemma 2.2])

$$P^\pm(u,v):=\frac{1}{2}(u\pm v,v\pm u). \tag{3.4}$$

Define the linear operator $L:E\to E$ as follows:

$$L(u,v):=(v,u). \tag{3.5}$$

If we write $z:=(u,v)\in E$ as $z=z^++z^-$ with $z^\pm\in E^\pm$, then

$$Lz=z^+-z^-.$$

Consider the functional $G:E\to\mathbf{R}$ defined by

$$G(z):=\frac{1}{2}(Lz,z)_E-\int_{\partial\Omega}H(x,u,v):=||z^+||_E-||z^-||_E-\mathscr{G}(z). \tag{3.6}$$

Then, under our assumptions, G is of $\mathbf{C}^1$ and $\mathscr{G}'$ is compact (cf. [BPR1, Theorem 2.1]). The derivative of G is given by

$$\begin{aligned}(G'(u,v),(\phi,\psi))_E \\ &= \langle A^{1/2}u, A^{1/2}\psi\rangle + \langle A^{1/2}\phi, A^{1/2}v\rangle - \int_{\partial\Omega} H_u(x,u,v)\phi \\ &\quad - \int_{\partial\Omega} H_v(x,u,v)\psi. \end{aligned} \tag{3.7}$$

Moreover, the critical point of G corresponds to the weak solution of (3.1)–(3.2) belonging to $W^{2,2}(\Omega)\times W^{2,2}(\Omega)$ (cf. [BPR2, Theorem 2.5]).

In order to prove Theorems 3.1–3.2, we intendto use Theorem 2.1 established in Section 2. Therefore, we consider the following family of functionals:

$$\begin{aligned}G_\lambda(z) &:= \lambda\frac{1}{2}||z^+||_E^2 - \frac{1}{2}||z^-||_E^2 - \int_{\partial\Omega} H(x,u,v) \\ &:= \frac{1}{2}(L_\lambda z, z)_E - \mathscr{G}(z), \quad \lambda\in[1,2],\end{aligned} \tag{3.8}$$

where

$$L_\lambda z = \lambda z^+ - z^-. \tag{3.9}$$

Lemma 3.1. *Assume that* (T_1)–(T_2) *hold. Then there exist* $C_0>0$ *such that* $G_\lambda(z)\geqslant -C_0$ *for all* $z\in E^+$ *and for all* $\lambda\in[1,2]$.

Proof. For any $z=(u,u)\in E^+$, then $||z||_E^2 = 2||u||^2_{E^{1/2}} = 2\langle A^{1/2}u, A^{1/2}u\rangle$ and

$$\langle A^{1/2}u, A^{1/2}u\rangle \geqslant \bar{\lambda}_1\int_{\partial\Omega} u^2. \tag{3.10}$$

By conditions (T_1),

$$H(x,u,u)\leqslant\bar{\lambda}_1(1+|u|^2), \quad \forall (x,u)\in\partial\Omega\times\mathbf{R}.$$

Therefore,

$$\int_{\partial\Omega} H(x,z)\leqslant\bar{\lambda}_1(|\partial\Omega| + ||u||^2_{L^2(\partial\Omega)}). \tag{3.11}$$

Combining (3.10)–(3.11), we have that

$$\begin{aligned}G_\lambda(\bar{z}) &\geqslant \lambda||z||_E^2 - C_0 - \bar{\lambda}_1||z||^2_{L^2(\partial(\Omega)} \\ &\geqslant -C_0\end{aligned}$$

for all $\lambda\in[1,2]$, where C_0 are constants independent of λ. $\square$

Lemma 3.2. *Assume that* (T_1)–(T_2) *hold. Then there exists* $R>0$ *such that* $G_\lambda(z) \leqslant -2C_0$ *uniformly for* $\lambda \in [1,2]$ *and* $z \in E^-, ||z||_E = R$.

Proof. In fact, for $z = (u, -u) \in E^-, ||z||_E = R$, we have

$$\begin{aligned} G_\lambda(z) &:= -\frac{1}{2}||z||_E^2 - \int_{\partial\Omega} H(x, u, -u) \\ &\leqslant -\frac{1}{2}||z||_E^2 = -\frac{1}{2}R^2 \leqslant -2C_0. \qquad \square \end{aligned}$$

Lemma 3.3. *Assume that* (T_1)–(T_2) *hold. Then there exists a sequence* $\{w_n\} \in E$ *and* $\lambda_n \in [1,2]$ *such that* $\lambda_n \to 1, G_{\lambda_n}{}'(w_n) = 0$.

Proof. This is a straightforward consequence of Lemmas 3.1–3.2, Example 2.1 and Theorem 2.1. $\square$

Lemma 3.4. *The sequence* $\{w_n\}$ *of Lemma* 3.3 *is bounded in* E.

Proof. Write $w_n := w_n^+ + w_n^- = (u_n, v_n)$. For any $\phi := (\varphi, \psi) \in E$, we have

$$\lambda_n(w_n^+, \phi)_E - (w_n^-, \phi)_E - \int_{\partial\Omega} (H_u(x, u_n, v_n)\varphi + H_v(x, u_n, v_n)\psi) = 0. \tag{3.12}$$

If $\{w_n\}$ is not bounded, we assume, up to a subsequence, that $||w_n||_E \to \infty$. Then

$$\frac{w_n}{||w_n||_E} \rightharpoonup \tilde{w} := (\tilde{u}, \tilde{v}), \quad \frac{w_n^+}{||w_n||_E} \rightharpoonup \tilde{w}^+, \quad \frac{w_n^-}{||w_n||_E} \rightharpoonup \tilde{w}^-, \tag{3.13}$$

weakly in E. By a simple computation, we have that

$$\frac{u_n}{||w_n||_E} \rightharpoonup \tilde{u}, \quad \frac{v_n}{||w_n||_E} \rightharpoonup \tilde{v}, \quad \text{weakly in } E^{1/2}.$$

Recall the compactness of the embedding $E^{1/2} \hookrightarrow L^2(\partial\Omega)$ (cf. [BPR1]), we see that

$$\frac{u_n}{||w_n||_E} \to \tilde{u}, \quad \frac{v_n}{||w_n||_E} \to \tilde{v}, \quad \text{strongly in } L^2(\partial\Omega). \tag{3.14}$$

Furthermore, by condition (T_2),

$$\left| \int_{\partial\Omega} \frac{H_u(x, u_n, v_n)\varphi + H_v(x, u_n, v_n)\psi}{||w_n||_E} \right| \to 0. \tag{3.15}$$

Combining (3.12)–(3.15), we get that

$$(\tilde{w}^+, \phi)_E - (\tilde{w}^-, \phi)_E = 0,$$

which implies that $\tilde{w}$ is a solution of the linear problem

$$\Delta u = u, \quad \Delta v = v \text{ in } \Omega; \quad \frac{\partial u}{\partial \eta} = 0, \ \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \partial\Omega.$$

If $\tilde{w} \neq 0$, we get a contradiction.
If $\tilde{w} = 0$, we write $w_n^+ := (x_n, x_n), w_n^- := (y_n, -y_n)$.
Since $(G_{\lambda_n}'(w_n), w_n^{\pm})_E = 0$, we have that

$$1 = \frac{\int_{\partial\Omega}(H_u(x, u_n, v_n)(x_n - \lambda_n y_n) + H_v(x, u_n, v_n)(x_n + \lambda_n y_n)}{\lambda_n ||w_n||_E^2}. \tag{3.16}$$

By assumption (T_2),

$$\int_{\partial\Omega} \frac{(H_u^2(x, u_n, v_n) + H_v^2(x, u_n, v_n))}{||w_n||_E^2} \leqslant c. \tag{3.17}$$

Moreover, $\tilde{w} = 0$, and the compactness of the imbedding implies that

$$\frac{x_n}{||w_n||_E} \to 0, \quad \frac{y_n}{||w_n||_E} \to 0 \quad \text{strongly in } L^2(\partial\Omega),$$

it follows that

$$\int_{\partial\Omega} \frac{|(x_n - \lambda_n y_n, x_n + y_n)|^2}{||w_n||_E^2} \to 0. \tag{3.18}$$

Evidently, (3.17)–(3.18) contradict Eq. (3.16). Therefore, $\{w_n\}$ is bounded in E. The rest is standard. $\square$

4. Application (II)

Many elliptic semilinear problems can be described in the following way. Let Ω be a domain in $\mathbf{R}^n$ (*bounded or unbounded*), and let $\mathscr{A}$ be a self-adjoint operator on $L^2(\Omega)$. We assume that $\mathscr{A} \geqslant \beta_0 > 0$ and that $\mathscr{C}_0^\infty(\Omega) \subset D := D(\mathscr{A}^{1/2}) \subset H^{m,2}(\Omega)$ for some $m > 0$, where $\mathscr{C}_0^\infty(\Omega)$ denotes the set of test functions in Ω (i.e., infinitely differentiable functions with compact supports in Ω) and $H^{m,2}(\Omega)$ denotes the Sobolev space. If m is an integer, the norm in $H^{m,2}(\Omega)$ is given by

$$||u||_{m,2} := \left(\sum_{|\mu| \leqslant m} ||D^\mu u||_2^2 \right)^{1/2}.$$

Here D^μ represents the generic derivative of order $|\mu|$ and the norm $|| \cdot ||_2$ on the right-hand side of is that of $L^2(\Omega)$. If m is not an integer, there are several ways of defining the space $H^{m,2}(\Omega)$, all of which are equivalent. We shall not assume that m is an integer.

Let q be a number satisfying

$$\begin{cases} 2<q\leqslant 2n/(n-2m), & 2m<n, \\ 2<q<\infty, n\leqslant 2m \end{cases} \tag{4.1}$$

and let $f(x,t)$ be a Carathéodory function on $\Omega\times\mathbf{R}$. This means that $f(x,t)$ is continuous in t for a.e. $x\in\Omega$ and measurable in x for every $t\in\mathbf{R}$.

In this subsection, we consider a system. Let $\mathscr{A}\geqslant\beta_0>0$, $\mathscr{B}\geqslant\mu_0>0$ be self-adjoint operators on $L^2(\Omega)$ with compact resolvents, where $\Omega\subset\mathbf{R}^N$ is not necessarily bounded, $\beta_0(\mu_0)$ is the lowest eigenvalue of $\mathscr{A}(\mathscr{B})$. Assume that the eigenfunctions of $\beta_0(\mu_0)$ are not equal to zero a.e. on Ω. Moreover, we assume that

$$\begin{cases} ||u||_q\leqslant c||\mathscr{A}^{1/2}u||_2 & \text{for all } u\in D(\mathscr{A}^{1/2}), \\ ||u||_q\leqslant c||\mathscr{B}^{1/2}u||_2 & \text{for all } u\in D(\mathscr{B}^{1/2}), \end{cases} \tag{4.2}$$

$||\cdot||_q$ is the usual norm in $L^q(\Omega)$. This restriction on constant q is reasonable by the Sobolev inequality.

Let $F(x,s,t), f(x,s,t), g(x,s,t)$ be Carathéodory functions on $\Omega\times\mathbf{R}^2$ satisfying

$$f(x,s,t)=\frac{\partial F}{\partial s}, \quad g(x,s,t)=\frac{\partial F}{\partial t}.$$

Assume

$$F(x,s,t)\geqslant 0, |f(x,s,t)|+|g(x,s,t)|\leqslant c(|s|+|t|+1), \quad \forall s,t\in\mathbf{R}, \ \forall x\in\Omega.$$

Here and elsewhere, the letter c will be indiscriminately used to denote various positive constants where the exact values are irrelevant. We wish to solve the system

$$\begin{cases} \mathscr{A}v=-f(x,v,w), \\ \mathscr{B}w=\beta g(x,v,w). \end{cases} \tag{S_β}$$

Remark 4.1. System (S_β) is a *partial* eigenvalue problem which demands a parameter depending functional as that in Theorem 2.1. In particular, the operator L_λ of Theorem 2.1 must be chosen carefully so that it can split into $\lambda L^{(1)}-L^{(2)}$, and the critical points of G_λ correspond to the solutions of (S_β). It seems that (S_β) have not been studied previously by classical critical point theory.

We obtained the following main results.

Theorem 4.1. *Assume that*

$$f(x,ty,0)/t\to\alpha_+(x)v^+(x)-\alpha_-(x)v^-(x), \quad \textit{as } t\to+\infty, \ y\to v,$$

where $a^{\pm} = \max\{\pm a, 0\}$. *Moreover,*

$$\alpha_{\pm}(x) \geqslant \not\equiv -\beta_0, \quad 2F(x,0,t) \leqslant \mu_0 t^2 + W(x), \quad x \in \Omega, \ \ t \in \mathbf{R},$$

where $W(x) \in L^1(\Omega)$. *Then system* (S_β) *has infinitely many solutions* (β, v, w).

Theorem 4.2. *Assume that*

$$f(x,0,tz)/t \to \delta_+(x)w^+(x) - \delta_-(x)w^-(x), \quad \text{as } t \to +\infty, \ \ z \to w$$

where $a^{\pm} = \max\{\pm a, 0\}$. *Moreover,*

$$\delta_{\pm}(x) \leqslant \not\equiv \mu_0, \quad 2F(x,s,0) \geqslant -\beta_0 s^2 - W(x), \quad x \in \Omega, \ \ t \in \mathbf{R},$$

where $W(x) \in L^1(\Omega)$. *Then system* (S_β) *has infinitely many solutions* (β, v, w).

Remark 4.2. Theorem 2.1 works perfectly in the proofs of Theorems 4.1 and 4.2. Further, by the assumptions of Theorems 4.1 and 4.2, we do not know whether or not there is a Palais–Smale compactness condition. Therefore, the classical linking theory cannot be applied directly. On the other hand, since for any given linking sets A, B, we cannot separate $\inf_B G_\lambda$ and $\sup_A G_\lambda$ by a positive constant, therefore, the linking theorem of [SZ] cannot be applied here.

Next, we consider the following assumptions:

(T_1) $F(x,0,t) = o(|t|^2)$ as $t \to 0$ uniformly for $x \in \Omega$.

(T_2) $F(x,s,t) = \frac{1}{2}m_0(|s|^2 + |t|^2) + K(x,s,t) \geqslant 0$ for all (x,s,t), where $m_0 > 2\mu_0$ and $K(x,s,t) = o(|s|^2 + |t|^2)$ as $|s| + |t| \to \infty$ uniformly for $x \in \Omega$.

We have

Theorem 4.3. *Assume* (T_1) *and* (T_2). *Then system* (S_β) *has infinitely many solutions* (β, v, w).

The assumptions in Theorems 4.1–4.3 which guarantee the existence of infinitely many solutions are quite weak. We are not aware of similar results elsewhere. Unfortunately, these theorems do not give any information for any specific β. We then turn our attention to solving (S_β) with $\beta = 1$. Some other conditions are needed of course. For $z = (s,t)$, we write $f(x,s,t) = f(x,z), g(x,s,t) = g(x,z), F(x,s,t) = F(x,z), F_z(x,z) = (\partial F/\partial s, \partial F/\partial t)$.

(T_3) $\liminf_{|z| \to 0} \frac{F_z(x,z)z}{F(x,z)} \geqslant \gamma_0 > 2$ uniformly for $x \in \Omega$.

(T_4) $\liminf_{|z| \to \infty} \frac{F_z(x,z)z - 2F(x,z)}{|z|^\alpha} \geqslant c > 0$ uniformly for $x \in \Omega$, where $\alpha \in (0,2)$.

(T_5) $F_z(x,z)z - 2F(x,z) > 0$ uniformly for $x \in \Omega$ and $|z| \neq 0$.
(T_6) $\liminf_{|z|\to 0} \frac{F(x,z)}{|z|^q} \geqslant c > 0$, $\limsup_{|z|\to 0} \frac{|g(x,z)|}{|z|^{q-1}} \leqslant c$, where q comes from (4.1).

Theorem 4.4. *Assume that* (T_1)–(T_6). *Then system* (S_1) *has a nontrivial solution* (v, w).

Remark 4.3. Assumptions (T_3), (T_4) and (T_6) are local conditions at both 0 and ∞. The hypotheses of both Theorems 4.1 and 4.2 do not necessarily imply the Palais–Smale condition. An asymptotically linear homoclinic orbit problem of Hamiltonian system or problem with jumping nonlinearity were studied in [SZ,SZo] where the assumptions are quite different from those here.

Remark 4.4. When $\beta = 1$, in paper [S2] (S_β) was studied where the abstract theory depends on the (PS) compactness condition. This required strong conditions. For example, the author of [S2] assumed that the system

$$-\mathscr{A}v = \alpha_+ v^+ - \alpha_- v^- + \beta_+ w^+ - \beta_- w^-$$

$$\mathscr{B}w = \gamma_+ v^+ - \gamma_- v^- + \delta_+ w^+ - \delta_- w^-$$

has only solution: $v = w = 0$, where $\alpha_\pm, \beta_\pm, \gamma_\pm, \delta_\pm$ are limit functions of f, g defined specifically. This is difficult to verify. On the other hand, when $\beta \neq 1$, the methods of [S2] do not appear to be applicable directly to the eigenvalue problem (S_β).

Remark 4.5. A typical example for operators $\mathscr{A}$ and $\mathscr{B}$ is $\mathscr{A} = -\Delta + V_1(x)$ and $\mathscr{B} = -\Delta + V_2(x)$ defined on $L^2(\mathbf{R}^N)$, where $V_i(x) \to \infty$ as $|x| \to \infty, i = 1, 2$ (cf. [Ra]). As for nonlinear term f (or g), we may just consider $f(x, s, t) = a|s| + b|t|$ with an appropriate choice of a, b according to the eigenvalues of $\mathscr{A}$ and $\mathscr{B}$.

Let $E = D(\mathscr{A}^{1/2}) \times D(\mathscr{B}^{1/2})$. Then E becomes a Hilbert space with norm given by $||u||^2 = (\mathscr{A}v, v) + (\mathscr{B}w, w), u = (v, w) \in E$. Define

$$G_\lambda(u) = \lambda b(w) - a(v) - 2\int_\Omega F(x, v, w)\, dx, \quad u \in E, \;\; \lambda \in [1, 2],$$

where $a(v) = (\mathscr{A}v, v), b(w) = (\mathscr{B}w, w)$. Then $G_\lambda \in \mathbf{C}^1(E, \mathbf{R})$ and

$$(G_\lambda{}'(u), h)/2 = \lambda(\mathscr{A}w, h_2) - (\mathscr{B}v, h_1) - (f(u), h_1) - (g(u), h_2),$$

$h = (h_1, h_2) \in E$. We write $f(u), g(u)$ in place of $f(x, v, w), g(x, v, w)$, respectively. It is readily seen that $G_\lambda{}'(u) = 0$ is equivalent to the systems (S_β) with $\beta = 1/\lambda$.

Let $E^- = \{(v,0) : (v,0) \in E\}, E^+ = \{(0,w) : (0,w) \in E\}$. Then E^-, E^+ are orthogonal closed subspaces such that $E = E^- \oplus E^+$. If we define $L_\lambda u = 2(-v, \lambda w)$ for all $u = (v,w) \in E, \lambda \in [1,2]$, then $L_\lambda = \lambda L^{(1)} - L^{(2)}$ is invertible self-adjoint bounded operator on E for all $\lambda \in [1,2]$, where $L^{(1)}u = 2(0,w), L^{(2)}u = 2(v,0)$. Also $G_\lambda{}'(u) = L_\lambda u + H'(u)$, where $H'(u) = -(\mathscr{A}^{-1}f(u), \mathscr{B}^{-1}g(u))$ is compact on E.

Proof of Theorem 4.1. We check that G_λ has the linking structure with respect to linking sets of Example 2.1. For $(0,w) \in E^+$, we see that

$$G_\lambda(0,w) \geqslant \lambda b(w) - \mu_0 ||w||_2^2 - \int_\Omega W(x)\, dx,$$

Therefore, $\inf_{E^+} G_\lambda \geqslant -\int_\Omega W(x)\, dx$ for all $\lambda \in [1,2]$.

We claim $\sup_{E^- \cap \partial B_R} G_\lambda \to -\infty$ as $R \to \infty$ uniformly for $\lambda \in [1,2]$. We follow the arguments of [S2]. Let $(v_k, 0)$ be any sequence in E^- such that $\rho_k^2 = a(v_k) \to \infty$. Then

$$G_\lambda(v_k, 0)/\rho_k^2 = -a(\bar{v}_k) - 2\int_\Omega F(x, v_k, 0)\, dx/\rho_k^2,$$

where $\bar{v}_k = v_k/\rho_k$. Since $a(\bar{v}_k) = 1$, there is a renamed subsequence $\bar{v}_k \to \bar{v}$ weakly in E^-, strongly in $L^2(\Omega)$ and a.e. in Ω such that

$$\begin{aligned} &G_\lambda(v_k, 0)/\rho_k^2 \\ &\quad \to -1 - \int_\Omega (\alpha_+(x)(\bar{v}^+(x))^2 + \alpha_-(x)(\bar{v}^-(x))^2)\, dx \\ &\quad = -\int_\Omega ((\beta_0 + \alpha_+(x))(\bar{v}^+(x))^2 + (\beta_0 + \alpha_-(x))(\bar{v}^-(x))^2)\, dx + \beta_0 ||\bar{v}(x)||_2^2 - 1. \end{aligned}$$

This is less than zero unless $\beta_0 ||\bar{v}||_2^2 = 1$. Since $a(\bar{v}) \leqslant 1$, this would mean that $\bar{v} \in E(\beta_0)$, the eigenspace of β_0. Thus $\bar{v} \neq 0$ a.e. by hypothesis. But then the integral cannot vanish since $\alpha_\pm \geqslant \not\equiv -\beta_0$. Hence

$$\limsup_{a(v) \to \infty} G_\lambda(0,v)/a(v) < 0 \quad \text{uniformly for } \lambda \in [1,2].$$

Our claim is true. By Theorem 2.1 and Example 2.1, G_λ has a critical point for almost all $\lambda \in [1,2]$. $\square$

Proof of Theorem 4.2. We interchange the roles of E^+ and E^- in the proof of Theorem 4.1. $\square$

Proof of Theorem 4.3. We show that G_λ satisfies all the conditions of the Theorem 2.1 with respect to the linking of Example 2.2.

By (T_1), for any $\varepsilon > 0$, there exists $c > 0$ such that $F(x, 0, w) \leqslant \varepsilon |w|^2 + c|w|^q$, where $q > 2$ comes from (4.1). Therefore, for any $u = (0, w) \in E^+$, we have that

$$\begin{aligned} G_\lambda(u) &= \lambda b(w) - 2\int_\Omega F(x, 0, w)\,dx \\ &\geqslant b(w) - 2\varepsilon ||w||_2^2 - 2c||w||_q^q \\ &\geqslant c \end{aligned}$$

for $||u||$ small enough and all $\lambda \in [1, 2]$.

Next, we choose $w_0 \neq 0$ such that $\mathscr{B} w_0 = \mu_0 w_0$. Define

$$A := \partial\{u = u^- + su_0 : u^- \in E^-, ||u|| \leqslant R, R > 0, s \geqslant 0\}, \text{ where } u_0 = (0, w_0).$$

We claim that $G_\lambda|_A \leqslant 0$ for some $R > 0$ for all $\lambda \in [1, 2]$. Note that $G_\lambda(u^-) \leqslant 0$ for all $u^- \in E^-$, if the claim is not true, then there is a sequence $u_n = s_n u_0 + u_n^-$ such that $||u_n|| \to \infty$ and $G_\lambda(u_n) > 0$. We write $u_n = (v_n, s_n w_0)$. Then

$$\lambda b(s_n w_0) - a(v_n) \geqslant 2\int_\Omega F(x, v_n, s_n w_0)\,dx \geqslant 0.$$

Since $||u_n||^2 = b(s_n w_0) + a(v_n)$, we may assume that $\frac{b(s_n w_0)}{||u_n||^2} \to s^* b(w_0)$. Then $s^* > 0$. Furthermore, note that

$$\begin{aligned} &2b(w_0) - m_0 \int_\Omega |w_0|^2\,dx \\ &\quad = 2\mu_0 \int_\Omega |w_0|^2\,dx - m_0 \int_\Omega |w_0|^2\,dx \\ &\quad < 0. \end{aligned}$$

Thus there exists a bounded set Ω_0 such that $2b(w_0) - m_0 \int_{\Omega_0} |w_0|^2 < 0$. Then,

$$\begin{aligned} 0 &\leqslant \frac{\lambda b(s_n w_0)}{||u_n||^2} - \frac{a(v_n)}{||u_n||^2} - \frac{2}{||u_n||^2}\int_\Omega F(x, v_n, s_n w_0)\,dx \\ &\leqslant \frac{2b(s_n w_0)}{||u_n||^2} - \frac{a(v_n)}{||u_n||^2} - \frac{2}{||u_n||^2}\int_{\Omega_0} F(x, v_n, s_n w_0)\,dx \\ &\leqslant \frac{2b(s_n w_0)}{||u_n||^2} - \frac{a(v_n)}{||u_n||^2} \\ &\quad - \frac{2}{||u_n||^2}\int_{\Omega_0} \left(\frac{1}{2} m_0 (|v_n|^2 + |s_n w_0|^2) + K(x, v_n, s_n w_0)\right) dx \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{2b(s_n w_0)}{\|u_n\|^2} - \frac{m_0}{\|u_n\|^2}\int_{\Omega_0} |s_n w_0|^2\, dx - \frac{2}{\|u_n\|^2}\int_{\Omega_0} K(x, v_n, s_n w_0)\, dx \\
&\to 2s^* b(w_0) - m_0 s^* \int_{\Omega_0} |w_0|^2\, dx \\
&< 0,
\end{aligned}
$$

providing a contradiction. Therefore, G_λ has the linking structure of Theorem 2.1 and has a critical point for almost all $\lambda \in [1, 2]$. This completes the proof of Theorem 4.3. $\square$

Proof of Theorem 4.4. By Theorem 4.3, we find two sequences $\{\lambda_n\}, \{u_n\}$ such that

$$\lambda_n \to 1, \quad G_{\lambda_n}{}'(u_n) = 0.$$

Moreover, note that since $\Gamma(s)u = su \in \Phi^*$, we have

$$
\begin{aligned}
G_{\lambda_n}(u_n) &= \inf_{\Gamma \in \Phi^*} \sup_{u \in A, s \in [0,1]} G_{\lambda_n}(\Gamma(s)u) \\
&\leqslant \sup_{u \in A, s \in [0,1]} G_{\lambda_n}(su) \\
&\leqslant c,
\end{aligned}
$$

where c is a constant independent of λ.

Now we claim that $\{u_n\}$ is bounded. Write $u_n = (v_n, w_n)$, then

$$\lambda_n b(w_n) - a(v_n) - 2\int_\Omega F(x, u_n) \leqslant c \tag{4.3}$$

and

$$(G_{\lambda_n}{}'(u_n), u_n) = \lambda_n b(w_n) - a(v_n) - \int_\Omega (f(x, u_n)v_n + g(x, u_n)w_n)\, dx = 0. \tag{4.4}$$

By (T_3)–(T_6), there is a $R_0 > 0$ such that

$$f(x, s, t)s + g(x, s, t)t \geqslant \gamma_0 F(x, s, t), \quad \forall x \in \Omega, |t| + |s| \leqslant R_0,$$

$$f(x, s, t)s + g(x, s, t)t - 2F(x, s, t) \geqslant c(|s|^2 + |t|^2)^{\alpha/2}, \quad \forall x \in \Omega, |t| + |s| \geqslant R_0,$$

$$F(x, s, t) \geqslant c(|s| + |t|)^q, \quad \forall x \in \Omega, |t| + |s| \leqslant R_0,$$

$$|g(x, s, t)| \leqslant c(|s| + |t|)^{q-1}, \quad \forall x \in \Omega, |t| + |s| \leqslant R_0.$$

Therefore,

$$
\begin{aligned}
c &\geqslant \left(\int_{|v_n|+|w_n|\leqslant R_0} + \int_{|v_n|+|w_n|\geqslant R_0} \right) (f(x,u_n)v_n + g(x,u_n)w_n - 2F(x,u_n))\,dx \\
&\geqslant \int_{|v_n|+|w_n|\leqslant R_0} (\gamma_0 - 2)F(x,v_n,w_n)\,dx + c\int_{|v_n|+|w_n|\geqslant R_0} (|v_n|^2 + |w_n|^2)^{\alpha/2}\,dx \\
&\geqslant \int_{|v_n|+|w_n|\leqslant R_0} (|v_n| + |w_n|)^q\,dx + \int_{|v_n|+|w_n|\geqslant R_0} (|v_n|^\alpha + |w_n|^\alpha)\,dx. \qquad (4.5)
\end{aligned}
$$

Since $(G_{\lambda_n}{}'(u_n),(0,w_n)) = \lambda_n b(w_n) - \int_\Omega g(x,v_n,w_n)w_n = 0$, we observe that

$$
\begin{aligned}
&\lambda_n b(w_n) \\
&\quad = \int_\Omega g(x,v_n,w_n)w_n \\
&\quad = \int_{|v_n|+|w_n|\geqslant R_0} g(x,v_n,w_n)w_n\,dx + \int_{|v_n|+|w_n|\leqslant R_0} g(x,v_n,w_n)w_n \\
&\quad \leqslant c\int_{|v_n|+|w_n|\geqslant R_0} (|v_n| + |w_n|)|w_n|\,dx \\
&\qquad + \int_{|v_n|+|w_n|\leqslant R_0} (|v_n| + |w_n|)^{q-1}|w_n|\,dx. \qquad (4.6)
\end{aligned}
$$

Therefore, choose $\bar{s} = \frac{q(2-\alpha)}{2(q-\alpha)}$, then $\bar{s}<1$. By (4.4) and (4.2),

$$
\begin{aligned}
&\int_{|v_n|+|w_n|\geqslant R_0} |w_n|^2 \\
&\quad = \int_{|v_n|+|w_n|\geqslant R_0} |w_n|^{2(1-\bar{s})}|w_n|^{2\bar{s}}\,dx \\
&\quad \leqslant \left(\int_{|v_n|+|w_n|\geqslant R_0} |w_n|^\alpha\,dx \right)^{2(1-\bar{s})/\alpha} \left(\int_{|v_n|+|w_n|\geqslant R_0} |w_n|^q\,dx \right)^{2\bar{s}/q} \\
&\quad \leqslant c||w_n||^{2\bar{s}} \\
&\quad = cb^{\bar{s}}(w_n) \qquad (4.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{|v_n|+|w_n|\leqslant R_0} (|v_n| + |w_n|)^{q-1}|w_n|\,dx \\
&\quad \times \left(\int_{|v_n|+|w_n|\leqslant R_0} (|w_n| + |v_n|)^q\,dx \right)^{(q-1)/q} \left(\int_{|v_n|+|w_n|\leqslant R_0} |w_n|^q\,dx \right)^{1/q} \\
&\quad \leqslant c. \qquad (4.8)
\end{aligned}
$$

Similarly,

$$\int_{|v_n|+|w_n|\geqslant R_0} |v_n|^2 \leqslant ca^{\bar{s}}(v_n). \tag{4.9}$$

By (4.3)–(4.8),

$$a(v_n) \leqslant \lambda_n b(w_n) \leqslant cb^{\bar{s}}(w_n) + ca^{\bar{s}}(v_n) + c.$$

This implies that $\{||u_n||\}$ is bounded since $\bar{s}$ is less than 1. Once this is done, we can use the usual procedures to show that there is a renamed subsequence such that $u_n \to u^*$ in E and u^* is a nontrivial solution of (S_1). $\square$

Acknowledgments

The authors thank the referee for careful reading the manuscript and for his/her useful suggestions and comments. Zou thanks UCIrvine for an appointment for the years 2001–2004.

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